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^{3}He-A_1 defects crossing the A_1–A_2 phase boundary

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Abstract. The fate of topologically stable defects in the ^{3}He-A_1 phase, which exists in a magnetic field, is analysed when this phase goes over to the A_2 phase at lower temperatures. It is shown that in magnetic fields that are not too large the A_1 monopoles become confined in the A_2 phase. It is argued that the Mermin–Ho relation breaks down in the presence of vortons.

1. Introduction

At zero magnetic field there are two superfluid ^{3}He phases: ^{3}He-A with symmetry $U(1)^S \times U(1)^{L-\Phi}$, and ^{3}He-B with symmetry $SO(3)^{S+L}$. These residual symmetry groups are subgroups of

$$G = SO(3)^S \times SO(3)^L \times U(1)^\Phi \tag{1.1}$$

which is the symmetry group of normal liquid ^{3}He. Depending on the pressure, P, the normal Fermi liquid may undergo a phase transition in the B phase either directly ($P < 21$ bar) or via the A phase ($P > 21$ bar). The transitions from the normal to the B and A phases are second-order transitions. The transition from the A to the B phase is a first-order transition.

On applying a magnetic field the A and B phases go over to extended A and B phases, the so-called A_2 and B_2 phases, respectively. The symmetry of the A_2 phase is $C_2^{S-\Phi} \times U(1)^{L-\Phi}$ [1], and the symmetry of the B_2 phase is $U(1)^{S+L}$, which is the U(1) subgroup of the symmetry group of ^{3}He-B, consisting of rotations about the magnetic field axis. The A_2–B_2 transition is, like the A–B transition, of first order. With increasing magnetic field the A_2 regime increases at the expense of the B_2 phase. Another feature of applying a magnetic field is the appearance of the ^{3}He-A_1 phase between the A_2 phase and the normal phase (N). The symmetry group of the A_1 phase is the group $U(1)^{S-\Phi} \times U(1)^{L-\Phi}$, and both the transitions N–A_1 and A_1–A_2 are of second order.

In the broken phases involved, there exists a variety of topologically stable defects. The purpose of this paper is to analyse defect transmutation in the sequence N $\rightarrow$ A_1 $\rightarrow$ A_2 of second-order phase transitions. We do not include the transition A_2–B_2 as it is of first order.

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However, before entering in a discussion of defect transformation in sections 3 and 4, we first elaborate on objects in the A_1 phase that look like true point defects, but that are not topologically stable. There are also similar objects in the ^{3}He-A phase. There they are called vortons. They were first introduced by Blaha [2], see also [3]. Contrary to the claim made in [2], these objects are topologically trivial [4]. In section 2, we clarify the crucial difference in topological nature between vortons and true monopoles (point defects) and comment on the validity of the so-called Mermin–Ho relation [5] in this context. To facilitate the discussion we introduce the language of non-Abelian gauge theories. Those interested only in the survival criteria of topologically stable defects crossing through a phase boundary could skip the following section.

2. Vortons in ^{3}He-A_1

For convenience, let us first consider the ^{3}He-A case. The ordering matrix in this instance has the value [6]

$$a^{(0)}_{m_S m_L} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

where m_S and m_L are the spin and orbital magnetic quantum numbers respectively, $m_S, m_L = 1, 0, -1$. In dyadic notation (2.1) becomes

$$a_{m_S m_L} = \hat{v}^3_{m_S}\, \hat{e}^+_{m_L} \tag{2.2}$$

where $\hat{e}^+ = -(1/\sqrt{2})(\hat{e}^1 + i\hat{e}^2)$, with $\hat{e}^1, \hat{e}^2, \hat{e}^3 = \hat{e}^1 \times \hat{e}^2$ a set of orthonormal vectors in orbital space, and $\hat{v}^3$ a unit vector in spin space. (The vector $\hat{e}^3$ is usually denoted by $\boldsymbol{l}$ and is called the orbital vector.) The residual symmetry of the A phase is

$$\mathrm{H} = \mathrm{U}(1)^S \times \mathrm{U}(1)^{L-\Phi} \tag{2.3}$$

where $\mathrm{U}(1)^{L-\Phi}$ denotes the group of simultaneous rotations about the $\hat{e}^3$ axis and gauge transformations. It is easily checked that the ordering matrix (2.2) is indeed invariant under this combined action:

$$\mathrm{U}(1)^{L-\Phi}: \quad a_{m_S m_L} \to a'_{m_S m_L} = \mathrm{e}^{\mathrm{i}(m_L \Lambda - \Lambda)} a_{m_S m_L} = a_{m_S m_L}. \tag{2.4}$$

The factor $\exp(-\mathrm{i}\Lambda)$ in (2.4) is the compensating gauge transformation. An important observation made by Volovik in this context [7] is that the transformation variable Λ of the residual $\mathrm{U}(1)^{L-\Phi}$ symmetry may depend on spacetime without spoiling the invariance of the theory. This follows trivially from the fact that under local $\mathrm{U}(1)^{L-\Phi}$ transformations

$$\begin{aligned} \partial_i a_{m_S m_L} \to \partial_i a'_{m_S m_L} &= \mathrm{i}(m_L \partial_i \Lambda(x) - \partial_i \Lambda(x)) \mathrm{e}^{\mathrm{i}(m_L \Lambda(x) - \Lambda(x))} a_{m_S m_L} \\ &\quad + \mathrm{e}^{\mathrm{i}(m_L \Lambda(x) - \Lambda(x))} \partial_i a_{m_S m_L} = \partial_i a_{m_S m_L} \end{aligned} \tag{2.5}$$

since $m_L = 1$. Therefore, the gradient terms in the Ginzburg–Landau functional will also be invariant under local $\mathrm{U}(1)^{L-\Phi}$ transformations. Hence, the $\mathrm{U}(1)^{L-\Phi}$ symmetry is a *local* gauge symmetry. Note, however, that there is no gauge field associated with the local symmetry. This is similar to the case of CP_N models which also possess a local U(1) gauge invariance without having a dynamical gauge field. Notwithstanding this fact, one may introduce a subsidiary field here that formally plays the role of a gauge field, and that makes explicit the local gauge invariance [8]. We will see that also in the case of ^{3}He-A one may introduce a subsidiary gauge field.

To obtain the Abelian connection we will employ the fact that the ordering matrix $a_{m_S m_L}$ transforms as a vector under $SO(3)^L$. Let us first consider the slightly more general case of an order parameter ψ which transforms under $SO(3)^L$ according to some arbitrary unitary representation. We write

$$\psi(x) = g(x)\psi_0 \tag{2.6}$$

with $g(x) \in SO(3)^L$ and ψ_0 some constant value. We have thus expressed the value of $\psi(x)$ at every point in spacetime as some rotation of the constant value ψ_0. The rotation matrix $g(x)$ differs from point to point, i.e. the spacetime dependence of $\psi(x)$ is introduced via the transformation $g(x)$. This can always be done in this way provided the norm of $\psi(x)$ is spacetime independent. This assumption is valid for weakly inhomogeneous states (the London limit). Next, we take the exterior derivative of (2.6)

$$\mathrm{d}\psi(x) = \mathrm{d}g\,\psi_0 = \mathrm{d}g\,g^{-1}\psi(x) \tag{2.7}$$

where $\mathrm{d}g\,g^{-1}$ is the so-called right-invariant Maurer–Cartan form. Since $\mathrm{d}g\,g^{-1}$ is an element of the Lie algebra it can be expanded in terms of the Hermitian generators T_A^L $(A = 1, 2, 3)$ of the $so(3)^L$ algebra. Therefore

$$\mathrm{d}\psi = \mathrm{i}E^A T_A^L \psi \tag{2.8}$$

with E^A one-forms

$$E^A = E^A{}_\mu\,\mathrm{d}x^\mu. \tag{2.9}$$

The representation of the generators T_A^L is fixed by the representation according to which ψ transforms. Equation (2.8) describes infinitesimal rotations with E^A the infinitesimal rotation angle. This equation may be cast in the form of a covariant derivative operating on ψ:

$$\mathrm{D}\psi = 0 \tag{2.10}$$

with $\mathrm{D} := \mathrm{d} - \mathrm{i}E^A T_A^L$, where the field E^A can now be interpreted as an $SO(3)^L$ gauge connection. On account of the identity

$$\mathrm{d}(\mathrm{d}g\,g^{-1}) - \mathrm{d}g\,g^{-1} \wedge \mathrm{d}g\,g^{-1} = 0 \tag{2.11}$$

it follows that the corresponding field strength F^A is zero, where

$$F^A := \mathrm{d}E^A + \tfrac{1}{2}\varepsilon_{ABC}\,E^B \wedge E^C \tag{2.12}$$

with $\mathrm{i}\varepsilon_{ABC}$ the structure constants of the $so(3)^L$ algebra:

$$[T_A^L, T_B^L] = \mathrm{i}\varepsilon_{ABC}\,T_C^L. \tag{2.13}$$

In components, the condition $F^A = 0$ is

$$F^A_{\mu\nu} = \partial_\mu E^A{}_\nu - \partial_\nu E^A{}_\mu + \varepsilon_{ABC} E^B{}_\mu E^C{}_\nu = 0 \tag{2.14}$$

where we used the antisymmetry of the structure constants.

Let us now focus on the case of superfluid ^{3}He-A. Since the ordering matrix, which transforms according to the adjoint representation under $SO(3)^L$, represented in the dyadic notation (2.2) involves the dreibein $\hat{e}^1, \hat{e}^2, \hat{e}^3$, it follows from (2.8) that the spatial derivative of $\hat{e}^3$ is given by†

$$\partial_i \hat{e}^3_A = -\varepsilon_{ABC} E^B{}_i \hat{e}^3_C \tag{2.15}$$

and similar equations for $\hat{e}^1$ and $\hat{e}^2$. Equation (2.15) has a general solution which is locally of the form

$$E^A_i = -\varepsilon_{ABC} \hat{e}^3_B \partial_i \hat{e}^3_C + a_i \hat{e}^3_A \tag{2.16}$$

where the Abelian connection one-form $a = \hat{e}^3_A E^A$ denotes the infinitesimal rotation angle associated with infinitesimal rotations about the $\hat{e}^3$ vector.

In fact, the U(1) gauge connection a thus obtained is the one that is relevant for the description of ^{3}He-A. For instance, the well known expression for the superfluid momentum $\boldsymbol{p}_s$ [7] can be cast in a form such that it explicitly contains a_i:

$$p_{s,i} = \tfrac{1}{2}(\partial_i \phi + a_i) \tag{2.17}$$

where ϕ is the transformation variable of the gauge group $U(1)^\Phi$. The usual form is obtained by writing a_i in terms of $\hat{e}^1$ and $\hat{e}^2$; we have

$$a_i = \hat{e}^1 \cdot \partial_i \hat{e}^2 \tag{2.18}$$

by virtue of (2.15) with the superscript 3 replaced by 2. Since under local $U(1)^{L-\Phi}$ transformations

$$\phi \to \phi - \Lambda \qquad a_i \to a_i + \partial_i \Lambda \tag{2.19}$$

expression (2.17) clearly brings out the fact that $\boldsymbol{p}_s$, being a physical observable, is invariant under these local transformations.

The local expression for the Abelian field strength f_{ij}

$$f_{ij} = \partial_i a_j - \partial_j a_i \tag{2.20}$$

follows directly from (2.14); we find that the field strength is in general non-zero:

$$f_{ij} = \hat{e}^3 \cdot (\partial_i \hat{e}^3 \times \partial_j \hat{e}^3). \tag{2.21}$$

By introducing the 'magnetic' field $b_i := \frac{1}{2}\varepsilon_{ijk} f_{jk}$, equation (2.21) with (2.20) yields the well known Mermin–Ho relation [5]

$$b_i = \tfrac{1}{2}\varepsilon_{ijk} \hat{e}^3 \cdot (\partial_j \hat{e}^3 \times \partial_k \hat{e}^3). \tag{2.22}$$

† The field E^A_i is minus the tensor field Ω^A_i introduced by Mermin and Ho [5].

Now, let us consider the point defects first stubied by Blaha [2]. Although they are not topologically stable as such, they may be made to become so by imposing the proper boundary conditions. Because of the tendency of the orbital rotation vector $\hat{e}^3$ to orient itself perpendicular to the boundary of the vessel containing the liquid, a possible choice is a spherical vessel [9]. With the point defect located at the origin one obtains the 'hedgehog' solution $\hat{e}^3 = \hat{r}$. A possible choice of the two other vielbeins is: $\hat{e}^1 = \hat{\varphi}$ and $\hat{e}^2 = \hat{\theta}$, where $\hat{\varphi}$ and $\hat{\theta}$ are unit vectors in the spherical coordinate system. The associated vector potential is of Dirac type

$$\boldsymbol{a} = \frac{\cos\theta}{r\sin\theta}\hat{\varphi} \tag{2.23}$$

with two string singularities at $\theta = 0$ and $\theta = \pi$, respectively. These vortex lines, each of unit flux ($= 2\pi$), are the famous Dirac strings. However, unlike in the case with genuine magnetic monopoles these strings are physical, and hence $U(1)^{L-\Phi}$ gauge invariant. In [9], Mermin gave a topological argument involving Euler's characteristic for the existence of these line singularities in the spherical vessel.

To illustrate the invariance of the strings take $\phi = 0$ and let $\boldsymbol{a}$ be given by (2.23), then

$$\boldsymbol{p}_s = \tfrac{1}{2}\boldsymbol{a} \tag{2.24}$$

and $\boldsymbol{p}_s$ has two line singularities. Next we perform a local $U(1)^{L-\Phi}$ transformation with transformation variable $\Lambda = \varphi$ where φ is the azimuth angle in real space. In this way the vector potential becomes

$$\boldsymbol{a} = \frac{\cos\theta + 1}{r\sin\theta}\hat{\varphi} \tag{2.25}$$

which has a single vortex line of two flux quanta emanating from the north pole. But the choice $\Lambda = \varphi$ introduces a string of unit flux in the ϕ-field, since $\phi \to \phi - \Lambda$ under local $U(1)^{L-\Phi}$ transformations. The resulting expression for the observable superfluid momentum has again two string singularities of unit flux at $\theta = 0$ and $\theta = \pi$, respectively; see figure 1. Hence the strings do indeed appear in a gauge invariant, observable quantity.

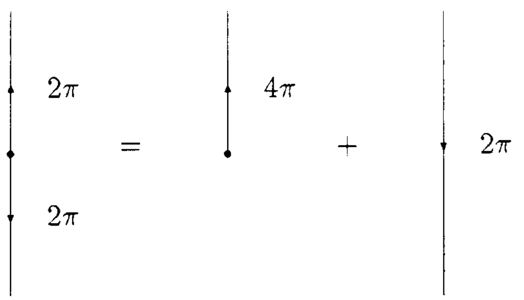

Figure 1. Gauge invariance of the vortex lines in the superfluid momentum field.

This fact, that the string singularity is physical, encourages us to take a closer look at the Mermin–Ho relation and consider its global properties. By integrating the right-hand side of (2.22) over the surface of the spherical vessel containing the liquid, with $\hat{e}^3 = \hat{r}$, we obtain 4π, while the left-hand side, assuming a_j is globally defined on the surface, will vanish by virtue of Stokes's theorem. Consequently, the Mermin–Ho relation (2.21) cannot be valid everywhere. We can imagine two possibilities. The first one is that the right-hand side is correct everywhere; then a_i describes a gauge field associated with a non-trivial U(1) fibre bundle over the sphere and (2.20) is only valid locally. This is so because the integral of the magnetic field over the two-sphere corresponds to the first Chern number which is non-zero. Therefore, we have to define two coordinate patches on the sphere; the relation (2.20) then holds within each coordinate patch, and has to be integrated locally. This would correspond with the Wu–Yang prescription for a genuine (magnetic) monopole field [10].

The other possibility, which is the correct one for the case at hand, is that we interpret the left-hand side of the Mermin–Ho relation globally in which case the right-hand side of the equation is incomplete. If we take, for example, the aforementioned Dirac potential, it is clear that a string singular term has to be added. This term should, of course, also arise from a careful calculation using (2.18). Indeed, we find

$$f_{ij} = (\partial_i\phi\partial_j\theta - \partial_j\phi\partial_i\theta)\sin\theta + 2\pi\varepsilon_{ij3}\cos\theta\delta(x)\delta(y) \tag{2.26}$$

where the second term is the singular term

$$2\pi\varepsilon_{ij3}\cos\theta\delta(x)\delta(y) = \pm 2\pi\varepsilon_{ij3}\delta(x)\delta(y) \tag{2.27}$$

for the string along the positive and negative z axis, respectively. So, we see that in the presence of vortons the Mermin–Ho relation is not valid as such. A singular term has to be added to account for the (physical) string. The total magnetic charge of the vorton is consequently zero: the magnetic flux emanating from the point defect is supplied by the string.

Let us now consider ^{3}He-A$_1$. The symmetry group of this phase is the group $\mathrm{U}(1)^{S-\Phi} \times \mathrm{U}(1)^{L-\Phi}$ which are both *local* symmetry groups. The ordering matrix is given by [6]

$$a^{(0)}_{m_S m_L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.28}$$

or in dyadic notation

$$a_{m_S m_L} = \hat{v}^+_{m_S}\,\hat{e}^+_{m_L} \tag{2.29}$$

where $\hat{v}^+ = -(1/\sqrt{2})(\hat{v}^1 + \mathrm{i}\hat{v}^2)$, with $\hat{v}^1, \hat{v}^2, \hat{v}^3 = \hat{v}^1 \times \hat{v}^2$ a set of orthonormal vectors in spin space. As the superfluid momentum [7]

$$p_{s,i} = \tfrac{1}{2}(\partial_i\phi + \hat{\mathbf{v}}^1 \cdot \partial_i\hat{\mathbf{v}}^2 + \hat{\mathbf{e}}^1 \cdot \partial_i\hat{\mathbf{e}}^2) \tag{2.30}$$

is a physical quantity, it is invariant under local $\mathrm{U}(1)^{S-\Phi}$ as well as under local $\mathrm{U}(1)^{L-\Phi}$ transformations. To make this apparent one can introduce gauge potentials $\boldsymbol{a}^S$ and $\boldsymbol{a}^L$ for the two local gauge groups, and write:

$$p_{s,i} = \tfrac{1}{2}(\partial_i\phi + a_i^S + a_i^L). \tag{2.31}$$

Mutatis mutandis this case may be analysed in the same way as the ^{3}He-A case. Suffice it to say that the string singularities in $\boldsymbol{a}^S$, $\boldsymbol{a}^L$ and ϕ may cancel in such a way that the corresponding superfluid momentum field has no vortex lines.

3. Defect transmutation

Our analysis of the fate of topologically stable objects in the A_1 phase when this phase goes over in the A_2 phase at lower temperatures is based on homotopy theory [11]. (For general accounts on homotopy theory see, for example, [12], [13] or [14].) In this section, we first briefly review [15] where one of us analyses how in general defects transmute when crossing a phase boundary.

Consider a system with symmetry G undergoing two subsequent second-order phase transitions $G \to H_1 \to H_2$. Without loss of generality we may assume that the group G is connected, i.e. $\pi_0(G) = 0$, and simply connected, i.e. $\pi_1(G) = 0$. The homotopy groups $\pi_2(G/H_1)$ and $\pi_1(G/H_1)$ denote the topological charges of respectively the point and line singularities in the H_1 phase, when this state is reached by breaking $G \to H_1$. Similarly, $\pi_i(G/H_2)$, with $i = 1, 2$, denote the charges of the point and line defects in the H_2 phase when we have the transition $G \to H_2$. However, since G is not directly broken to H_2, but via H_1, we cannot expect $\pi_1(G/H_2)$ to describe all line singularities in the H_2 phase. It may well be the case, for example, that point singularities that are present in the H_1 phase develop a line singularity in the second transition $H_1 \to H_2$. These line singularities are not accounted for by $\pi_1(G/H_2)$. On the other hand, it turns out that $\pi_2(G/H_2)$ describes all point defects in the H_2 phase, including those that descend from point defects in the H_1 phase.

First, we discuss what might happen with the line defects in the H_2 phase that descend from point defects in the H_1 phase. The argument is of a dynamical character and does not follow from topological considerations. In crossing the phase boundary between the two domains with H_1 and H_2 symmetry, the radial point-defect configuration is expected to bend over in a vortex-type configuration. That is, the flux of the point defect will be squeezed into a tube upon entering the H_2 phase. If the length of the tube is such that its total energy becomes of order twice the energy of the point defect, then it will be energetically favourable to create a pair of H_1 point defects with total charge zero. In this way, these point defects become confined in the H_2 phase. We will refer to the associated line defects as finite or open line defects as opposed to ordinary, infinite line singularities which may also be finite but closed. It is to be remarked here that it is not mandatory for the cores of these line defects in the H_2 phase to be in the normal state; they may be in the H_1 state.

The relevant information about the topological objects we are interested in is contained in the homotopy sequence [15]

$$0 \simeq \pi_2(H_1) \to \pi_2(H_1/H_2) \to \pi_2(G/H_2) \to \pi_2(G/H_1) \to \pi_1(H_1/H_2) \to \pi_1(G/H_2) \to \pi_1(G/H_1). \quad (3.1)$$

This sequence is exact, which means that the kernel (Ker) of each map is the image (Im) of the previous map. More specifically, we have the following isomorphisms:

$$0 \simeq \text{Ker}\,\{\pi_2(H_1/H_2) \to \pi_2(G/H_2)\} \quad (3.2)$$

$$\text{Im}\,\{\pi_2(H_1/H_2) \to \pi_2(G/H_2)\} \simeq \text{Ker}\,\{\pi_2(G/H_2) \to \pi_2(G/H_1)\} \quad (3.3)$$

$$\text{Im}\,\{\pi_2(G/H_2) \to \pi_2(G/H_1)\} \simeq \text{Ker}\,\{\pi_2(G/H_1) \to \pi_1(H_1/H_2)\} \quad (3.4)$$

$$\text{Im}\,\{\pi_2(G/H_1) \to \pi_1(H_1/H_2)\} \simeq \text{Ker}\,\{\pi_1(H_1/H_2) \to \pi_1(G/H_2)\} \quad (3.5)$$

$$\text{Im}\,\{\pi_1(H_1/H_2) \to \pi_1(G/H_2)\} \simeq \text{Ker}\,\{\pi_1(G/H_2) \to \pi_1(G/H_1)\}. \quad (3.6)$$

The isomorphism (3.2) implies that

$$\pi_2(H_1/H_2) \simeq \text{Im}\,\{\pi_2(H_1/H_2) \to \pi_2(G/H_2)\} \subset \pi_2(G/H_2) \quad (3.7)$$

and $\pi_2(G/H_2)$ therefore describes indeed all point singularities in the H_2 phase.

From (3.3) it follows that those monopoles in the H_2 phase that can also exist when, instead of G, H_1 is broken to H_2 have a trivial monopole charge in H_1. It also follows from (3.3) that all point defects in the H_2 phase not contained in $\text{Im}\,\{\pi_2(H_1/H_2) \to \pi_2(G/H_2)\}$ correspond to non-trivial elements of $\pi_2(G/H_1)$, i.e. these point defects descend from point defects that already exist in the H_1 phase. This statement is summarised as follows:

$$\begin{aligned}\pi_2(G/H_2)/\text{Im}\,\{\pi_2(H_1/H_2) \to \pi_2(G/H_2)\} &\simeq \pi_2(G/H_2)/\text{Ker}\,\{\pi_2(G/H_2) \to \pi_2(G/H_1)\}\\ &\simeq \text{Im}\,\{\pi_2(G/H_2) \to \pi_2(G/H_1)\}\end{aligned} \tag{3.8}$$

where the first isomorphism follows from the exactness of the sequence (3.1). We note that these point defects in the H_2 phase may have acquired other quantum numbers in the transition $H_1 \to H_2$, which pertain to the H_2 phase rather than to the H_1 phase.

The isomorphism (3.4) leads us to the conclusion that point singularities in the H_1 phase that are contained in $\text{Im}\,\{\pi_2(G/H_2) \to \pi_1(G/H_1)\}$ cannot develop a line singularity in the transition $H_1 \to H_2$. These point defects will therefore either survive the transition $H_1 \to H_2$ (possibly by converting their quantum numbers) or vanish. On the other hand, those point defects in the H_1 phase not contained in $\text{Im}\,\{\pi_2(G/H_2) \to \pi_1(G/H_1)\}$ will develop a (finite) line singularity in the phase transition $H_1 \to H_2$, as may be inferred from the following:

$$\begin{aligned}\pi_2(G/H_1)/\text{Im}\,\{\pi_2(G/H_2) \to \pi_2(G/H_1)\} &\simeq \pi_2(G/H_1)/\text{Ker}\,\{\pi_2(G/H_1) \to \pi_1(H_1/H_2)\}\\ &\simeq \text{Im}\,\{\pi_2(G/H_1) \to \pi_1(H_1/H_2)\}.\end{aligned} \tag{3.9}$$

The isomorphism (3.5) reveals that line singularities in the H_2 phase that originate from a monopole in the H_1 phase correspond to the trivial element of $\pi_1(G/H_2)$: these open line defects cannot exist when G is broken directly to H_2. By the same token, (3.5) shows that line defects in the H_2 phase that can exist when H_1 is broken to H_2, but that do not descend from a point defect in the H_1 phase, correspond to non-trivial elements of $\pi_1(G/H_2)$. In other words, these line defects are infinitely long line defects which have no analogue in the H_1 phase. They are therefore called *new* infinite line defects. As a formula we have

$$\begin{aligned}\pi_1(H_1/H_2)/\text{Im}\,\{\pi_2(G/H_1) \to \pi_1(H_1/H_2)\} &\simeq \pi(H_1/H_2)/\text{Ker}\,\{\pi_1(H_1/H_2) \to \pi_1(G/H_2)\}\\ &\simeq \text{Im}\,\{\pi_1(H_1/H_2) \to \pi_1(G/H_2)\}\end{aligned} \tag{3.10}$$

where we again used the exactness of the sequence (3.1).

Finally, we note that (3.6) contains the evident statement that the line defects in the H_2 phase that can also exist when, instead of G, H_1 is broken to H_2 correspond to the trivial element of $\pi_1(G/H_1)$, i.e. they have no line defect charge in H_1. Moreover, we infer from (3.6) that all elements of $\pi_1(G/H_2)$ not contained in $\text{Im}\,\{\pi_1(H_1/H_2) \to \pi_1(G/H_2)\}$ correspond to non-trivial elements of $\pi_1(G/H_1)$

$$\begin{aligned}\pi_1(G/H_2)/\text{Im}\,\{\pi_1(H_1/H_2) \to \pi_1(G/H_2)\} &\simeq \pi_1(G/H_2)/\text{Ker}\,\{\pi_1(G/H_2) \to \pi_1(G/H_1)\}\\ &\simeq \text{Im}\,\{\pi_1(G/H_2) \to \pi_1(G/H_1)\}.\end{aligned} \tag{3.11}$$

In physical terms this means that these line defects in the H_2 phase have an analogue in the H_1 phase and, consequently, they are infinite line defects. The quantum numbers of the line singularities under consideration may, however, have been converted in the transition $H_1 \to H_2$.

This ends our brief review of [15]. As an illustration, we consider a three-dimensional isotropic Heisenberg ferromagnet that undergoes two subsequent phase transitions. The symmetry G of the unbroken phase is the group of rotations $G = \mathrm{SO}(3)$. Below the Curie temperature the ferromagnet has a spontaneous magnetic moment thus breaking rotational invariance. The residual symmetry H_1 of this state is the group of rotations about the magnetisation axis, $H_1 = \mathrm{SO}(2) \simeq \mathrm{U}(1)$. Let us assume that at still lower temperatures there is a second phase transition, where the group U(1) is broken to $H_2 = \mathrm{C}_2$, with C_2 being the group of rotations by the angle π. Whereas the first transition may be achieved by giving a non-zero expectation value to a spin-1 field, for this second transition to happen a field with spin ≥ 2 must develop a non-zero expectation value. For convenience we will consider the simply connected covering group $\bar{G} = \mathrm{SU}(2)$ of SO(3) instead of SO(3) itself. This is not a severe limitation as we are mainly interested in the transition $\mathrm{U}(1) \to \mathrm{C}_2$. The relevant non-trivial homotopy groups for the case under consideration are: $\pi_2(\bar{G}/H_1) \simeq \mathrm{Z}$, indicating the possible existence of monopoles in the H_1 phase; $\pi_1(H_1/H_2) \simeq 2\mathrm{Z}$, with 2Z the set of even integers, showing that the (infinite and finite) line defects in the H_2 phase, when this state is reached by the transition from the H_1 phase, have an even charge; and $\pi_1(\bar{G}/H_2) \simeq \mathrm{Z}_2 \simeq \{0, 1\}$ which means that there is only one type of topologically non-trivial infinite line singularity in the H_2 phase. When two such line defects coalesce they simply annihilate each other, since $1 + 1 = 0$ in Z_2. Thus the sequence (3.1) takes the form

$$\begin{aligned} 0 \to \pi_2(H_1/H_2) \simeq 0 \to \pi_2(\bar{G}/H_2) \simeq 0 \to \pi_2(\bar{G}/H_1) \simeq \mathrm{Z} \\ \to \pi_1(H_1/H_2) \simeq 2\mathrm{Z} \to \pi_1(\bar{G}/H_2) \simeq \mathrm{Z}_2 \to \pi_1(\bar{G}/H_1) \simeq 0. \end{aligned} \tag{3.12}$$

Using the exactness of (3.12), we find that

$$\mathrm{Im}\ \{\pi_1(H_1/H_2) \to \pi_1(\bar{G}/H_2)\} \simeq \mathrm{Z}_2 \tag{3.13}$$

which shows that all infinitely long line defects of the H_2 phase are new line defects. This was to be expected as the H_1 phase supports no line singularities. A closer inspection of (3.13) reveals that Ker $\{\pi_1(H_1/H_2) \to \pi_1(\bar{G}/H_2)\} \simeq 4\mathrm{Z}$, with 4Z denoting the set $\{0, \pm 4, \pm 8, \pm 12, ...\}$. Consequently, we obtain the isomorphism

$$\mathrm{Im}\ \{\pi_2(\bar{G}/H_1) \to \pi_1(H_1/H_2)\} \simeq 4\mathrm{Z} \tag{3.14}$$

from which we infer that all point defects in the H_1 phase develop a (finite) line singularity in the transition $H_1 \to H_2$. More specifically, a point defect in the H_1 phase with charge n gives rise to a finite line defect with charge $4n$ in the H_2 phase.

In conclusion, in the H_1 phase with symmetry U(1) there can exist only point defects which upon entering the H_2 phase become confined. Besides the finite line singularities with charges labelled by elements of 4Z, the H_2 phase with symmetry C_2 also has infinite line singularities, the latter being characterised by an element of Z_2.

In the next section we discuss superfluid ^{3}He in a magnetic field. In particular, we study the fate of point defects in ^{3}He-A_1 when this phase goes over in the A_2 state. As it turns out, the situation is similar to the case of the ferromagnet we discussed above.

4. The $A_1 \to A_2$ transition

As before, we take instead of the SO(3) factors in (1.1) the covering groups $SU(2)^S$ and $SU(2)^L$, i.e. we replace G by

$$\bar{G} = SU(2)^S \times SU(2)^L \times U(1)^\Phi. \tag{4.1}$$

Here, we assumed that the magnetic field is not too large, so that the group $SO(3)^S$ is not explicitly broken to the group $U(1)^S$ of rotations about the magnetic field axis. The case of a large magnetic field will be discussed towards the end of this section. In this instance with

$$H_1 = U(1)^{S-\Phi} \times U(1)^{L-\Phi} \tag{4.2}$$

and

$$H_2 = C_2^{S-\Phi} \times U(1)^{L-\Phi} \tag{4.3}$$

we have the coset spaces

$$\bar{G}/H_1 \simeq (SU(2) \times SU(2))/S_1^{S-L} \tag{4.4}$$

$$\bar{G}/H_2 \simeq (SU(2) \times SU(2))/Z_2^{S-L} \tag{4.5}$$

and

$$H_1/H_2 \simeq U(1)^{S-\Phi}/C_2^{S-\Phi} \simeq P_1^{S-\Phi}(\mathbb{R}) \tag{4.6}$$

with the real projective space $P_1^{S-\Phi}(\mathbb{R})$ denoting the circle $S_1^{S-\Phi}$ with antipodal points identified. The coset (4.5) with $\bar{G}$ replaced by G was first given by Bailin and Love [16]. Since the factor $U(1)^{S-\Phi}$ in (4.2) is replaced by $C_2^{S-\Phi}$ in (4.3), only two antipodal points of the circle S_1^{S-L} in (4.4) survive in (4.5).

The coset (4.5) may also be inferred directly from the value of the ordering matrix. In the spherical bases the value for the A_2 phase is [6]†

$$a^{(0)}_{m_S m_L} = \begin{pmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix} \tag{4.7}$$

with $m_S, m_L = 1, 0, -1$, and r and x two non-zero arbitrary complex parameters. In dyadic notation (4.7) becomes

$$a_{m_S m_L} = r\hat{v}^+_{m_S}\hat{e}^+_{m_L} + x\hat{v}^-_{m_S}\hat{e}^+_{m_L} \tag{4.8}$$

where $\hat{v}^+ = -(1/\sqrt{2})(\hat{v}^1 + i\hat{v}^2)$ and $\hat{v}^- = (1/\sqrt{2})(\hat{v}^1 - i\hat{v}^2)$, with $\hat{v}^1, \hat{v}^2, \hat{v}^3 = \hat{v}^1 \times \hat{v}^2$ a set of orthonormal vectors in spin space. Similarly, $\hat{e}^+ = -(1/\sqrt{2})(\hat{e}^1 + i\hat{e}^2)$, with $\hat{e}^1, \hat{e}^2, \hat{e}^3 = \hat{e}^1 \times \hat{e}^2$ a set of orthonormal vectors in orbital space. It is now easily recognised that the coset space is the product of two SU(2) factors, which specify the orientation of the dreibein in spin and orbital space, respectively, factorised with

† In [6] we erroneously stated that the phase with symmetry $U(1)^{L-\Phi}$ is the A_2 phase. The correct symmetry is $C_2^{S-\Phi} \times U(1)^{L-\Phi}$ with (4.7) the corresponding value of the ordering matrix [1].

respect to the group† C_2^{S-L}. The factorisation reflects the observation that the ordering matrix (4.8) remains unchanged if one performs simultaneously a spin rotation about the $\hat{v}^3$ axis by an angle π, and an orbital rotation about the $\hat{e}^3$ axis by an angle $-\pi$. Under these operations $\hat{v}^{\pm} \to \exp(\pm i\pi)\hat{v}^{\pm}$, and $\hat{e}^{+} \to \exp(-i\pi)\hat{e}^{+}$. Because of the absence of the last term in (4.8) in the case of ^{3}He-A$_1$, see (2.29), the corresponding ordering matrix is invariant under an arbitrary simultaneous rotation about the $\hat{v}^3$ and $\hat{e}^3$ axis: $\hat{v}^{+} \to \exp(i\alpha)\hat{v}^{+}$, $\hat{e}^{+} \to \exp(-i\alpha)\hat{e}^{+}$. This explains the factor S_1 in (4.4). We note that the gauge group $U(1)^{\Phi}$ is not relevant for this analysis, since this group is intertwined with $SO(3)^S$ in the same manner as it is with $SO(3)^L$. Consequently, $\exp(i\phi) \in U(1)^{\Phi}$ factors cancel in the process of factorisation.

Having identified the coset spaces, the relevant homotopy groups follow directly. We have $\pi_2(\bar{G}/H_1) \simeq Z^{S-L}$, $\pi_1(\bar{G}/H_2) \simeq Z_2^{S-L}$, $\pi_1(H_1/H_2) \simeq 2Z^{S-\Phi}$, and $\pi_1(\bar{G}/H_1) \simeq \pi_2(\bar{G}/H_2) \simeq \pi_2(H_1/H_2) \simeq 0$. Consequently, the sequence (3.1) reduces to a form very much like the sequence (3.12)

$$\begin{aligned} 0 &\to \pi_2(H_1/H_2) \simeq 0 \to \pi_2(\bar{G}/H_2) \simeq 0 \to \pi_2(\bar{G}/H_1) \simeq Z^{S-L} \\ &\to \pi_1(H_1/H_2) \simeq 2Z^{S-\Phi} \to \pi_1(\bar{G}/H_2) \simeq Z_2^{S-L} \to \pi_1(\bar{G}/H_1) \simeq 0 \end{aligned} \tag{4.9}$$

and the conclusions are similar. More specifically, the A$_1$ phase supports no line defects, but it may have point defects that are topologically stable. The charges of these singularities being elements of Z^{S-L} can take any integer value. In the transition A$_1$ → A$_2$ the point defects develop a finite line singularity, and hence are confined in the A$_2$ phase. The charge of the finite line defects is $n^{S-\Phi} = 4n^{S-L} = 0, \pm 4, \pm 8, ...$, where n^{S-L} is the charge of the point defect from which the line singularity descends. In the A$_2$ phase there are also line defects that are infinitely long. As the fundamental homotopy group $\pi_1(\bar{G}/H_2) \simeq Z_2^{S-L}$, it follows that two infinite line defects with non-trivial charge annihilate each other when they coalesce. Besides the difference in charge the finite and infinite line defects in the A$_2$ phase differ also in the group spaces involved, as is evident from the superscripts of the charges: $S - \Phi$ for the finite and $S - L$ for the infinite line singularities.

We thus far assumed the system is not exposed to a strong magnetic field. For larger fields the symmetry group $SO(3)^S$ is explicitly broken to $U(1)^S$, the group of spin rotations about the magnetic field axis. In this case we have instead of (4.4) and (4.5) the coset spaces

$$\bar{G}/H_1 \simeq (U(1) \times SU(2))/S_1^{S-L} \tag{4.10}$$

and

$$\bar{G}/H_2 \simeq (U(1) \times SU(2))/Z_2^{S-L} \tag{4.11}$$

respectively. The coset space (4.6) remains unaltered. The relevant homotopy groups are given by $\pi_2(\bar{G}/H_1) \simeq \pi_1(\bar{G}/H_1) \simeq 0$, cf [16], and also by $\pi_2(\bar{G}/H_2) \simeq 0$, and $\pi_1(\bar{G}/H_2) \simeq 2Z$. From this we infer that a strong magnetic field destroys the point defects that are present in the A$_1$ phase in not too large magnetic fields. This can easily be understood, as a radial point defect configuration is not compatible with the

† It is obvious that the choice of sign in front of S and L in the superscript is arbitrary; the group may equivalently be denoted by C_2^{S+L}.

preferred direction introduced by a strong magnetic field. Because of the absence of point defects in the A_1 phase there are no finite line singularities in the A_2 phase. There are, however, still infinite line singularities in this phase which are labelled by elements of 2Z.

As pointed out by Salomaa [17], monopoles may occur also in a different context as free ends of line singularities, namely in rotating superfluid ^{3}He. More specifically, he argues that since vortices cannot perforate the A–B interface as such, a lattice of monopoles might materialise at this (rotating) phase boundary, each monopole being the endpoint of a vortex line in ^{3}He-A or ^{3}He-B. The density and charge of these monopoles will be such as to fulfil the macroscopic rotation condition in the plane of the interface.

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